Koopman-LQR Controller for Quadrotor UAVs from Data

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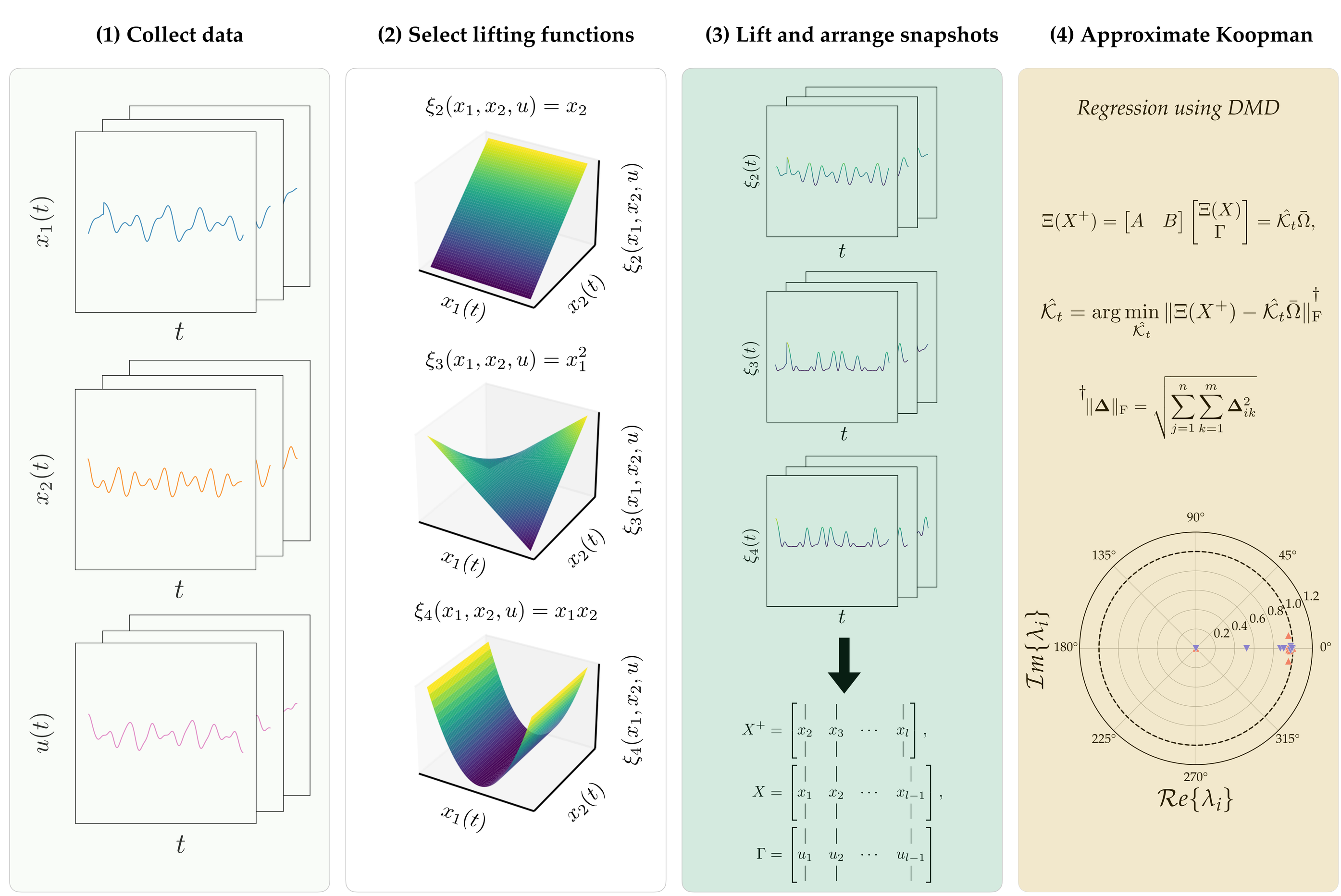
Main takeaway

Goal: To control highly unstable, nonlinear system with the simplicity inherent in the linear control with global linear approximation.

Desired properties for our scheme:

1. **Global Linearity**. Converts nonlinear quadrotor dynamics into a globally linear model instead of local traditional linearization (e.g. Taylor linearization).
2. **Efficiency**. Enables fast and computationally light control design.
3. **Stability**. Ensures robust stabilization of highly unstable systems.



We present, given in the figure above, the **first scheme to deal with all of these issues!**

Koopman operator

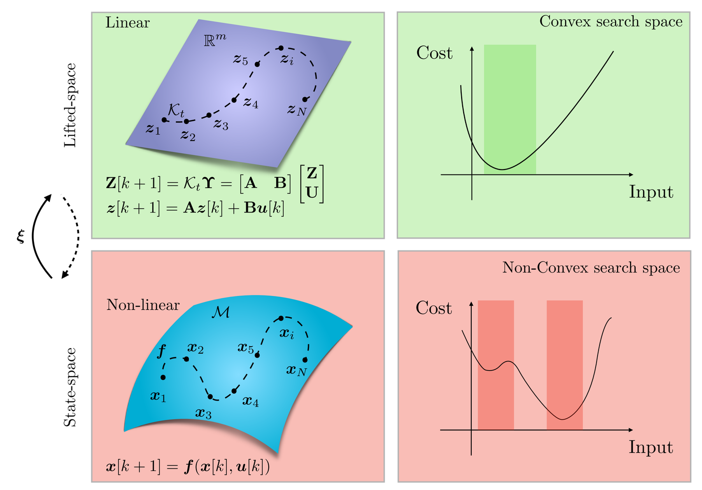
Koopman operator: consider the discrete time dynamical system:

$$x_k^+ = f(x_k, u_k),$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^l$ is the control input, f is a transition map, and x^+ is the successor state. The Koopman operator $\mathcal{K}_t$ is an infinite-dimensional operator:

$$\mathcal{K}_t \xi = \xi \circ f(x_k, u_k),$$

acting on $\xi \in \mathcal{H} : \mathbb{R}^n \times \mathbb{R}^l \mapsto \mathbb{R}$, where $\circ$ denotes function composition.



Koopman operator is an effective method to offer a linear representation of non-linear systems in in **infinite-dimensional space** by its action on the Hilbert space $\mathcal{H}$ of measurement functions ξ.

Extended dynamic mode decomposition (EDMD)

Koopman's infinite-dimensional nature requires a finite approximation. **EDMD** with the right observables achieves this.

Theorem: consider a dynamical system $x_k^+ = f(x_k, u_k) \approx Ax_k + Bu_k$ and dataset $\mathcal{D}$. The system can be written as:

$$\Xi(X^+) = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \Xi(X) \\ \Gamma \end{bmatrix} = \hat{\mathcal{K}}_t \bar{\Omega},$$

solving

$$\hat{\mathcal{K}}_t = \arg\min_{\hat{\mathcal{K}}_t} \|\Xi(X^+) - \hat{\mathcal{K}}_t \bar{\Omega}\|_F.$$

will get an approximation for the Koopman operator.

After resolving the operator $\hat{\mathcal{K}}_t$ the linear lifted approximation of nonlinear dynamics becomes

$$z_k^+ = Az_k + Bu_k$$
$$x_k = Cz_k.$$

with a higher dimension that the original state space of the system.

Motivated by the literature we come up with observable functions as

$$\Xi(x) = \begin{bmatrix} 1, & x, & p_{WB}, & \dot{p}_{WB}, & \sin(p_{WB}), & \cos(p_{WB}), & \text{vec}(R \times \omega_{WB}) \end{bmatrix} \in \mathbb{R}^{39}.$$

Koopman meets LQR

Consider a quadratic cost function:

$$\mathcal{J} = \underset{u_0,\ldots,u_{N-1}}{\text{minimize}} \sum_{0}^{N-1} x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau),$$

If Koopman linearization is considered, the cost function still holds but with minor modifications as follows

$$\bar{Q} = \begin{bmatrix} Q_{n\times n} & 0_{p-n\times p-n} \\ 0_{p-n\times p-n} & 0_{p-n\times p-n} \end{bmatrix}_{p\times p}, \quad \bar{R} = R$$

Theorem: It is possible to have a matrix L, and a control law in the form of $u = -Lx$ such that the cost $\mathcal{J}$ is minimum with a Koopman linearized dynamics.

Application to quadrotor

Koopman-LQR can be applied to control and stabilize very nonlinear topoligies like quadrotors.

Quadrotor application: Consider a quadrotor dynamics given by the following

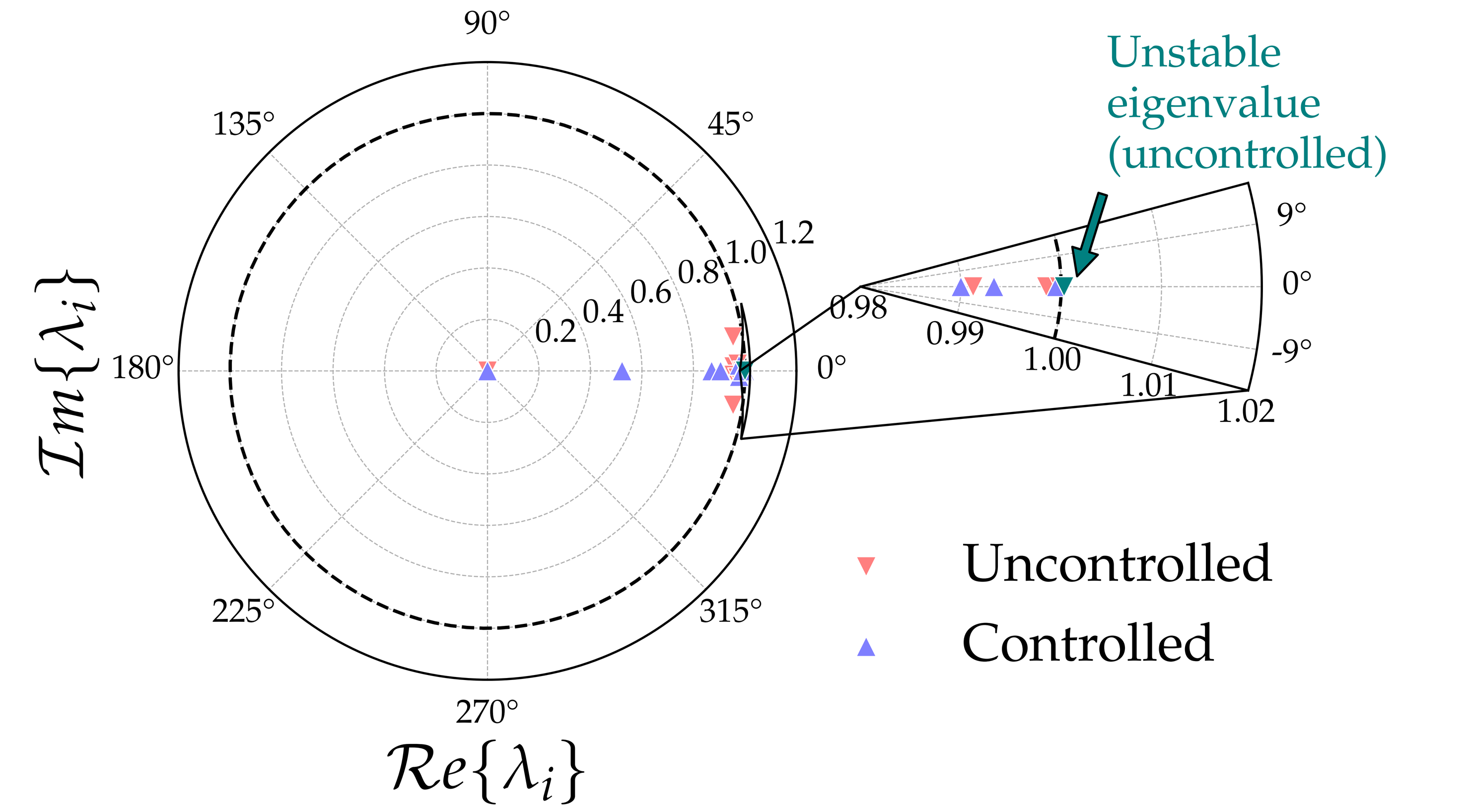
$$\dot{x} = \frac{d}{dt} \begin{bmatrix} p_{WB} \\ \dot{p}_{WB} \\ q_{WB} \\ \omega_B \end{bmatrix} = f(x, u) = \begin{bmatrix} p_W \\ \frac{1}{m} q_{WB} \cdot T_B + g_W \\ \frac{1}{2} q_{WB} \otimes \omega_B \\ J^{-1}(\tau_B - \omega_B \times J\omega_B) \end{bmatrix},$$

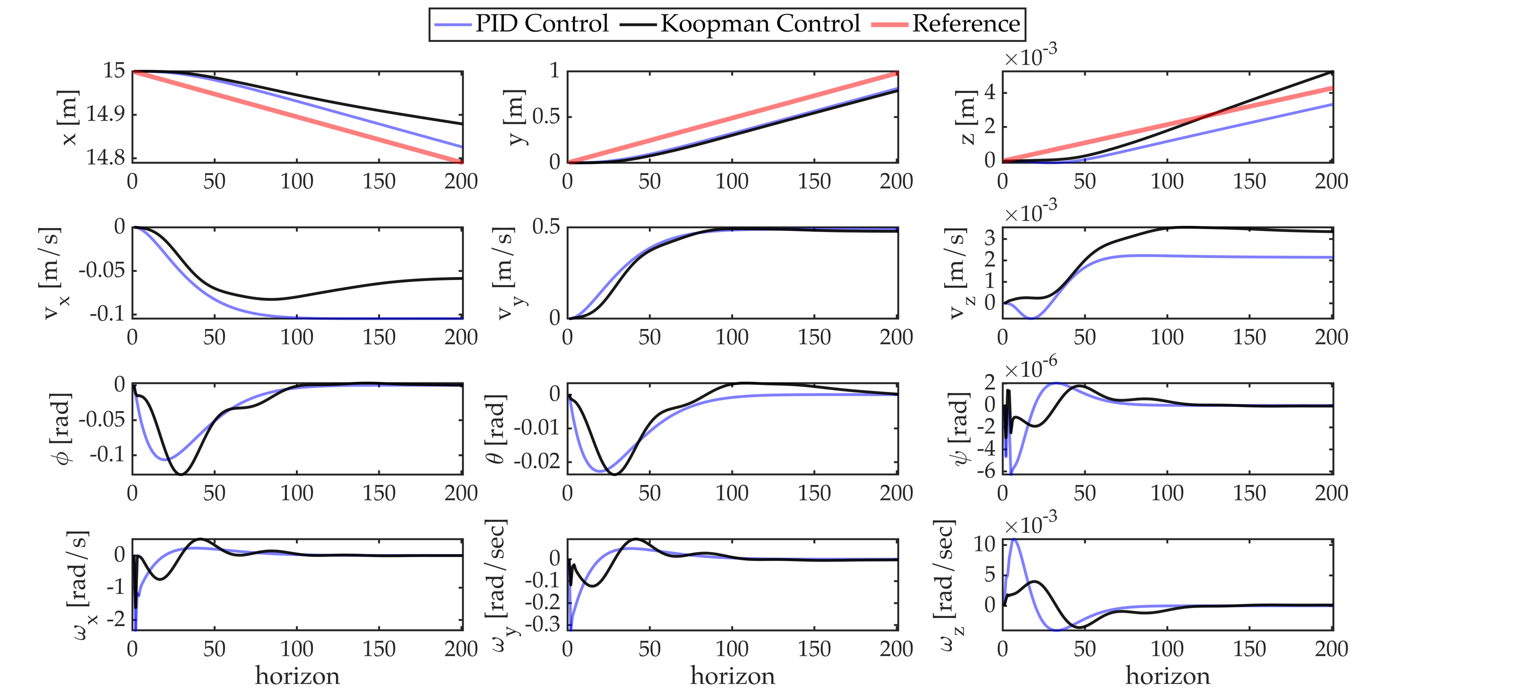
and

$$T_B = \begin{bmatrix} 0 \\ 0 \\ \sum T_i \end{bmatrix} \quad \text{and} \quad \tau_B = \begin{bmatrix} l(-T_0 - T_1 + T_2 + T_3) \\ l(-T_0 + T_1 + T_2 - T_3) \\ c_\tau(-T_0 + T_1 - T_2 + T_3) \end{bmatrix},$$

the goal is to derive a linear formulation of this system then stabilize it.

Application to quadrotor: stability and performance analysis





The problem uses **39 observable functions**. Learning the Koopman operator on an **M2 MacBook Air** took **0.1393 seconds**. Inference over **200 timesteps** totaled **0.0035 seconds**, averaging **1.74 × 10⁻⁵ seconds per iteration**.

states	%NRMSE
p_{WB} – Position	3.2529 ± 2.3216
$\dot{p}_{WB}$ – Velocity	4.8129 ± 3.8608
$\mathcal{E}_{WB}$ – Euler angles	2.4398 ± 2.4263
ω_B – Angular velocity	7.8525 ± 6.6367
Mean	4.5895 ± 3.8114

Check out our paper!

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